1.8 – Introduction to Linear Transformations

Definitions: $\mathbf{R}^2$ is the set of all ordered pairs, $\mathbf{R}^3$ is the set of all ordered triples, and $\mathbf{R}^n$ is the set of all ordered n-tuples. Elements of R^n are called **vectors**.

The **standard basis vectors** for R^n are $\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, $\ldots, \mathbf{e}_n = \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}$. These can also be written as row vectors if convenient.

Definitions: A **function** f is a rule that associates with an input (an element of the **domain** of f) a unique output (an element of the **codomain** of f). The output is the **image** of the input, and the set of all images is called the **range** of f. Note that the range of f is a subset of the codomain of f.

Definition: If T is a function with domain R^n and codomain R^m, then we say that T is a **transformation** from R^n to R^m or that T **maps** from R^n to R^m, which we denote by writing $T : R^n \to R^m$. In the special case where $m = n$, a transformation is sometimes called an **operator** on R^n.

#**5** Find the domain and codomain of the transformation defined by the matrix product.

a. $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Definitions: A **matrix transformation** $\mathbf{w} = A\mathbf{x}$ maps a vector $\mathbf{x} \in R^n$ to a vector $\mathbf{w} \in R^m$ by multiplying $\mathbf{x}$ on the left by A [which is an $m \times n$ matrix]. If $m = n$, then we call the transformation a **matrix operator**. A matrix transformation is denoted $T_A : R^n \to R^m$ or $\mathbf{w} = T_A(\mathbf{x})$ if we do not need to specify the domain and codomain. This can also be written in the form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

verbalized as "T_A maps $\mathbf{x}$ into $\mathbf{w}$." The matrix A is the **standard matrix** for the transformation.

#**8** Find the domain and codomain of the transformation T defined by the formula.

a. $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$

b. $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

#**12** Find the standard matrix for the transformation defined by the equations.

a.
$$\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2 \end{aligned}$$

b.
$$\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

#13 Find the standard matrix for the transformation T defined by the formula.

a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

b. $T(x_1, x_2, x_3) = (4x_1 + x_2, x_1 + x_2)$

Definitions: The **zero transformation** from R^n to R^m, $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$, maps every vector in R^n to the zero vector in R^m. The **identity operator** $T_{I_n}(\mathbf{x}) = I_n(\mathbf{x}) = \mathbf{x}$ maps every vector in R^n to itself.

Theorem 1.8.1 Properties of Matrix Transformations

For every matrix A the matrix transformation $T_A : R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

a) $T_A(\mathbf{0}) = \mathbf{0}$

b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

Theorem 1.8.2 $T : R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

i) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

ii) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

Definitions: The properties (i) and (ii) in the preceding theorem are called **linearity conditions**. A transformation that possesses these properties is a **linear transformation**.

Theorem 1.8.3 Every linear transformation from R^n to R^m is a matrix transformation, and conversely every matrix transformation from R^n to R^m is a linear transformation.

Theorem 1.8.4 If $T_A : R^n \rightarrow R^m$ and $T_B : R^n \rightarrow R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.

#22 Use Theorem 1.8.2 to show that T is a matrix transformation.

a. $T(x,\ y,\ z) = (x + y,\ y + z,\ x)$

b. $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

23. Use Theorem 1.8.2 to show that T is not a matrix transformation.

a. $T(x,\ y) = (x^2, y)$

b. $T(x,\ y,\ z) = (x,\ y,\ xz)$

#**28** The images of the standard basis vectors for R^3 are given for a linear transformation $T : R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Example: Find the standard matrix A for the linear transformation $T : R^2 \to R^3$ for which

$$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$$

and use it to compute $T\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right)$.

Matrix operators on R^2

Reflection operators

- Reflection about the x-axis: $T(x,\ y) = T(x,\ -y)$, $T_A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Reflection about the y-axis: $T(x,\ y) = T(-x,\ y)$, $T_A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Reflection about the line $y = x$: $T(x,\ y) = (y,\ x)$, $T_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Projection operators

- Orthogonal projection onto the x-axis: $T(x,\ y) = T(x,\ 0)$,
  $T_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Orthogonal projection onto the y-axis: $T(x,\ y) = T(0,\ y)$,
  $T_A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Rotation operator

- Counterclockwise rotation about the origin through an angle θ:
  $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Matrix operators on R^3

Reflection operators

- Reflection about the xy-plane: $T(x,\ y,\ z) = (x,\ y,\ -z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Reflection about the xz-plane: $T(x,\ y,\ z) = (x,\ -y,\ z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Reflection about the yz-plane: $T(x,\ y,\ z) = (-x,\ y,\ z)$,

$$T_A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Projection operators

- Orthogonal projection onto the xy-plane: $T(x,\ y,\ z) = (x,\ y,\ 0)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Orthogonal projection onto the xz-plane: $T(x,\ y,\ z) = (x,\ 0,\ z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Orthogonal projection onto the yz-plane: $T(x,\ y,\ z) = (0,\ y,\ z)$,

$$T_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$